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Article

On Bayesian Estimation for Modified-Weibull Distribution

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Abstract

This article studies the Bayesian estimation in newly developed Modified-Weibull (M-W) Distribution. The M-W distribution is suitable for bath-shaped and non-decreasing hazards. The bayes estimators of shape and scale parameters and reliability characteristics of the distribution are obtained under squared error loss function (SELF). Furthermore, the bayes estimates are comupted using Tierney Kadane approximation and Markov Chain Monte Carlo techniques. The High Posterior Density (HPD) Credible intervals of parameters are also obtained. A Monte Carlo simulation study is performed to compare the Maximum likelihood estimators and Bayes Estimators. Under certain circumstances, the proposed estimators are observed more efficient than already existing estimators. Finally, a real data set is used for illustrative purposes.

Keywords: *M-W Distribution, Maximum likelihood estimation, Bayes estimation, SELF, MCMC methods. M-H algorithm.*

1. Introduction

In reliability theory, the classical models were frequently used to model the data because of their interpretability and ease of mathematics. But there are variety of hazard rate behaviours that are seen in real-world scenarios, such as non-monotonic, bathtub-shaped or unimodal failure rates. The introduction of more adaptable and generalized distributions that can represent a greater variety of failure mechanisms within a single framework is motivated by these limitations. Therefore, the new lifetime distributions are developed to adequately represent the complexity of failure and survival data in the real world. For instance, Ghitney et al. (2013), Ijaz, Mashwani, & Belhaouari (2020), Muse et al. (2021), Nwezza & Ugwuowo (2020), Modi et al. (2020), Kumawat & Nagar (2024) and Mudholkar et al. (1993), Modi (2021) proposed some new models to deal with both monotonic and non-monotonic hazards effectively and provide results with superior flexibility over existing models. The Classical and Bayesian framework is used in drawing inferences about the parameters of these models by researchers. For example, Generalized Exponential Distribution Kundu & Gupta (2008), Kundu and Pradhan (2009), Kumarswamy Distribution Chaturvedi (2023), Inverse Weibull distribution Kundu & Howlader (2010), Singh et al. (2013), Inverse Rayleigh distribution Dey (2012) and Aslam et al. (2021) are studied under Bayesian framework. Similarly, several studies have applied Bayesian methods to get improved results over traditional methods.

In this article, the estimation procedure for newly developed Modified-Weibull (M-W) studied by Modi, Kumar, & Singh (2020), Kumawat, Modi & Nagar (2023) is discussed. The M-W

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distribution is a suitable model for many real-life situations where the data follows non-decreasing, bath-shaped, unimodal failure or hazard rate function.

The pdf of M-W distribution is expressed as follows:

$$f(x) = \frac{\alpha^{\beta} \frac{p}{\sigma^{p}} x^{p-1} e^{-\left(\frac{x}{\sigma}\right)^{p}}}{(1+\alpha^{\beta}) \left\{1 - \frac{e^{-\left(\frac{x}{\sigma}\right)^{p}}}{(1+\alpha^{\beta})}\right\}^{2}}, \tag{1}$$

where $x > 0$, $\alpha > 0$, $\beta > 0$, $p > 0$ and $\sigma > 0$. We note that (α, β) are modified family parameters and (p, σ) are shape and scale parameters respectively.
Moreover, the hazard function (HF) and survival function (SF) are given respectively as,

$$h(x) = \frac{\left(1+\alpha^{\beta}\right) \frac{p}{\sigma^{p}} x^{p-1}}{1+\alpha^{\beta} - e^{-\left(\frac{x}{\sigma}\right)^{p}}}, x > 0, \& \ \alpha, \beta, p, \sigma > 0 \tag{2}$$

$$S(x) = \frac{\alpha^{\beta} e^{-\left(\frac{x}{\sigma}\right)^{p}}}{1+\alpha^{\beta} - e^{-\left(\frac{x}{\sigma}\right)^{p}}}, \quad x > 0, \& \ \alpha, \beta, p, \sigma > 0 \tag{3}$$

The maximum likelihood (ML) estimation has been used earlier for estimation purposes in M-W distribution. Now, we study the bayesian inference for the parameters and reliability characteristics of the M-W distribution under SELF following the assumption that the modified family parameters are known. The SELF is used here which is symmetric and considers the overestimation and under estimation equally serious. The numerical techniques that are used to approximate the solutions are Tierney-Kadane (TK) and Markov Chain Monte Carlo (MCMC) technique using the Metropolis-Hastings (MH) algorithm. These techniques have severally used in Bastan & Mirmostafaee (2019) and Smith & Roberts (1993).
The outline of paper is summarized as: Section 2 contains the ML estimation of the parameters of M-W distribution in which ML estimates and the corresponding ACIs are derived. Section 3 considers the Bayes estimation of the parameters with HPD credible intervals. We compare the estimators by conducting a Monte Carlo simulation in section 4. Section 5 contains a detailed real data analysis with data visualization for illustrative purposes. Finally, section 6 contains the conclusion based on the study.

2. Maximum Likelihood Estimation

Let $X_1, X_2, \dots, X_n$ be a set of random observations of size n drawn from M-W distribution. Then, the log- likelihood function of the sample is given as,

$$\ln L(x;\alpha,\beta,p,\sigma) = n\ln p - np\ln\sigma + n\beta\ln\alpha - n\ln\left(1+\alpha^{\beta}\right) + (p-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{p} - 2\sum_{i=1}^{n}\ln\left(1-\frac{e^{-\left(\frac{x_i}{\sigma}\right)^{p}}}{(1+\alpha^{\beta})}\right) \tag{4}$$

Since α and β are known, the ML equations to obtain the estimates of p and σ are obtained as,

$$\frac{\partial \ln L}{\partial p} = \frac{n}{p} - n\ln\sigma + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{p}\ln\left(\frac{x_i}{\sigma}\right) - 2\sum_{i=1}^{n}\left(\left(\frac{x_i}{\sigma}\right)^{p}\ln\left(\frac{x_i}{\sigma}\right)\frac{e^{-\left(\frac{x_i}{\sigma}\right)^{p}}}{\left(1+\alpha^{\beta}-e^{-\left(\frac{x_i}{\sigma}\right)^{p}}\right)}\right) = 0 \tag{5}$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{np}{\sigma} + \frac{p}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{p} + 2\frac{p}{\sigma}\sum_{i=1}^{n}\left(\left(\frac{x_i}{\sigma}\right)^{p}\frac{e^{-\left(\frac{x_i}{\sigma}\right)^{p}}}{\left(1+\alpha^{\beta}-e^{-\left(\frac{x_i}{\sigma}\right)^{p}}\right)}\right) = 0 \tag{6}$$

It is clear from the equations (5) and (6) that the analytical solution of these equations is not possible to obtain. Therefore, we use numerical optimization techniques by Fletcher (1987) to obtain ML estimates in R software. Following the invariance property, the SF and HF can also be estimated using eq. (2) & (3).

2.2 Asymptotic Confidence Intervals (ACI)

The ACIs are derived using the asymptotic normal property of ML estimates. Let $\hat{\theta} = (\hat{p},\hat{\sigma})$ be the estimator of $\theta = (p, \sigma)$, then information matrix $I(\hat{\theta})$ is obtained as

$$I(\hat{\theta}) = -\left[I_{ij}\right]_{\hat{\theta}} = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial p^2} & -\frac{\partial^2 \ln L}{\partial p \partial \sigma} \\ -\frac{\partial^2 \ln L}{\partial \sigma \partial p} & -\frac{\partial^2 \ln L}{\partial \sigma^2} \end{bmatrix}_{(\hat{p},\hat{\sigma})}$$

Then, the approximate variance-covariance matrix of estimators defined by $\left[I(\hat{\theta})\right]^{-1}$ and the 100(1- δ)% ACIs of parameters are obtained as,

$\left(\hat{p} - Z_{\frac{\delta}{2}}\sqrt{Var(\hat{p})}, \hat{p} + Z_{\frac{\delta}{2}}\sqrt{Var(\hat{p})}\right)$ and $\left(\hat{\sigma} - Z_{\frac{\delta}{2}}\sqrt{Var(\hat{\sigma})}, \hat{\sigma} + Z_{\frac{\delta}{2}}\sqrt{Var(\hat{\sigma})}\right)$ respectively.

3. Bayesian estimation

In Bayesian estimation method, the parameters of the distribution are assumed to be random variables with a prior density function. Since parameters α and β are known, parameters p and σ are estimated in this section. The prior information for parameters can be obtained using the past knowledge and experiments. When the prior information is not available, we use non-informative priors for the estimation. Now, we assume that parameters p and σ have independent gamma priors with hyperparameters a_1, b_1 and a_2, b_2 respectively. The prior densities of parameters p and σ is given by,

$$\pi(p) = \frac{{b_1}^{a_1}}{\Gamma a_1} p^{a_1-1} e^{-b_1 p};\ p > 0;\ a_1, b_1 > 0 \tag{7}$$

$$\pi(\sigma) = \frac{{b_2}^{a_2}}{\Gamma a_2} \sigma^{a_2-1} e^{-b_2 \sigma};\ \sigma > 0;\ a_2, b_2 > 0 \tag{8}$$

the joint prior pdf of p and σ is,

$$\pi\,(p,\sigma) \propto\ p^{a_1-1}\sigma^{a_2-1}\, e^{-(b_1 p + b_2 \sigma)};\ p > 0, \sigma > 0 \tag{9}$$

Next, the joint posterior distribution of p and σ given x is given by:

$$\pi(p,\sigma\,|x)\ = \frac{L(x|\ p,\sigma)\pi(\ p,\sigma)}{\int_0^\infty \int_0^\infty L(x|\ p,\sigma)\pi\,(p,\sigma)\,dpd\sigma}$$

$$\pi(p,\sigma\,|x)\ = C^{-1} p^{n+a_1-1} \sigma^{-np+a_2-1}\, e^{-(b_1 p + b_2 \sigma)} e^{-\Sigma\left(\frac{x_i}{\sigma}\right)^p} \prod_{i=1}^{n} \frac{{x_i}^{p-1}}{\left\{1 - \frac{e^{-\left(\frac{x_i}{\sigma}\right)^p}}{(1+\alpha^\beta)}\right\}^2} \tag{10}$$

where,

$$C^{-1} = \int_0^\infty \int_0^\infty p^{n+a_1-1} \sigma^{-np+a_2-1} e^{-\Sigma\left(\frac{x_i}{\sigma}\right)^p} \prod_{i=1}^{n} \frac{{x_i}^{p-1}}{\left\{1 - \frac{e^{-\left(\frac{x_i}{\sigma}\right)^p}}{(1+\alpha^\beta)}\right\}^2} dpd\sigma$$

Therefore, Bayes estimator of any function of p and σ say, $\varphi\,(p,\sigma)$ under SELF is the posterior expectation of $\varphi\,(p,\sigma)$ and is given by,

$$E(\varphi(p,\sigma)\,|x) = \int_0^\infty \int_0^\infty \varphi(\ p,\sigma)\ \pi(p,\sigma\,|x)dp\ d\sigma \tag{11}$$

It seems that Eq. (13) is not in explicit form and its solution is not possible to obtain analytically. Therefore, the TK method by Tierney and Kadane (1986) and MH algorithm is used to approximate Bayes estimates.

3.1 TK Approximation

According to TK method, the approximate bayes estimator of any function φ (p, σ) under SELF is given by,

$$\widehat{\varphi} = E(\varphi(p,\sigma)|x) = \frac{\int_0^\infty \int_0^\infty e^{n\delta^*(p,\sigma)}\varphi(p,\sigma)\,dpd\sigma}{\int_0^\infty \int_0^\infty e^{n\delta(p,\sigma)}\varphi(p,\sigma)\,dpd\sigma} \tag{12}$$

where,

$$\delta(p,\sigma) = \frac{\ln \pi(p,\sigma) + \ln L}{n}$$

and $\delta^*(p,\sigma) = \delta(p,\sigma) + \frac{1}{n}\ln \varphi(p,\sigma)$.

The above expression is approximated by the T-K method and the posterior mean is computed as follows:

$$\widehat{\varphi}(p,\sigma) = E(\varphi(p,\sigma)|x) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}}e^{n\delta^*(\hat{p}_{\delta^*},\hat{\sigma}_{\delta^*})} - e^{n\delta(\hat{p}_\delta,\hat{\sigma}_\delta)} \tag{13}$$

where,

$$n\delta(p,\sigma) = (n + a_1 - 1)\ln p + (-np + a_2 - 1)\ln\sigma + n\beta\ln\alpha - n\ln\left(1+\alpha^\beta\right) - (b_1 p + b_2\sigma) + (p-1)\sum_{i=1}^n \ln x_i - \sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^p - 2\sum_{i=1}^n \ln\left(1 - \frac{e^{-\left(\frac{x_i}{\sigma}\right)^p}}{(1+\alpha^\beta)}\right)$$

Then, $(\hat{p}_\delta, \hat{\sigma}_\delta)$ are obtained by solving equations simultaneously:

$$\frac{\partial\delta(p,\sigma)}{\partial p} = \frac{n + a_1 - 1}{p} - n\ln\sigma - b_1 p + \sum_{i=1}^n \ln x_i - \sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^p \ln\left(\frac{x_i}{\sigma}\right) - 2\sum_{i=1}^n \left(\left(\frac{x_i}{\sigma}\right)^p \ln\left(\frac{x_i}{\sigma}\right)\frac{e^{-\left(\frac{x_i}{\sigma}\right)^p}}{\left(1+\alpha^\beta - e^{-\left(\frac{x_i}{\sigma}\right)^p}\right)}\right) = 0$$

$$\frac{\partial\delta(p,\sigma)}{\partial\sigma} = -\frac{np}{\sigma} - b_2\sigma + \frac{p}{\sigma}\sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^p + 2\frac{p}{\sigma}\sum_{i=1}^n \left(\left(\frac{x_i}{\sigma}\right)^p \frac{e^{-\left(\frac{x_i}{\sigma}\right)^p}}{\left(1+\alpha^\beta - e^{-\left(\frac{x_i}{\sigma}\right)^p}\right)}\right) = 0$$

and |Σ| is the determinant of inverse of negative Hessian of δ (p, σ) at $(\hat{p}_\delta, \hat{\sigma}_\delta)$ obtained as,

$$|\Sigma|=\begin{pmatrix} -\frac{\partial^2\delta(\mathrm{p},\sigma)}{\partial p^2} & -\frac{\partial^2\delta(\mathrm{p},\sigma)}{\partial p\partial\sigma} \\ -\frac{\partial^2\delta(\mathrm{p},\sigma)}{\partial\sigma\partial p} & -\frac{\partial^2\delta(\mathrm{p},\sigma)}{\partial\sigma^2} \end{pmatrix}^{-1}$$

Now, to obtain the bayes estimate of p and σ, we use $\varphi(\mathrm{p},\sigma) = \mathrm{p}$ and $\varphi(\mathrm{p},\sigma) = \sigma$ respectively to get

$$\delta_p^*(\mathrm{p},\sigma) = \delta\,(\mathrm{p},\sigma) + \frac{1}{\mathrm{n}}\ln \mathrm{p}$$

$$\delta_\sigma^*(\mathrm{p},\sigma) = \delta\,(\mathrm{p},\sigma) + \frac{1}{\mathrm{n}}\ln \sigma$$

Then, $(\hat{\mathrm{p}}_{\delta^*}, \hat{\sigma}_{\delta^*})$ are computed by solving equations simultaneously:

$$\frac{\partial\delta_{\mathrm{p}}^*(\mathrm{p},\sigma)}{\partial\mathrm{p}} = \frac{\partial\delta(\mathrm{p},\sigma)}{\partial\mathrm{p}} + \frac{1}{\mathrm{p}} = 0$$

$$\frac{\partial\delta_{\mathrm{p}}^*(\mathrm{p},\sigma)}{\partial\sigma} = \frac{\partial\delta(\mathrm{p},\sigma)}{\partial\sigma} + \frac{1}{\sigma} = 0$$

and Σ_φ^* is the determinant of inverse of negative Hessian of $\delta_\varphi^*(\mathrm{p},\sigma)$ at $(\hat{\mathrm{p}}_{\delta^*}, \hat{\sigma}_{\delta^*})$. $|\Sigma_\varphi^*|$ is obtained as

$$|\Sigma_{\mathrm{p}}^*|=\begin{pmatrix} -\frac{\partial^2\delta_\varphi^*(\mathrm{p},\sigma)}{\partial \mathrm{p}^2} & -\frac{\partial^2\delta_\varphi^*(\mathrm{p},\sigma)}{\partial \mathrm{p}\,\partial\sigma} \\ -\frac{\partial^2\delta_\varphi^*(\mathrm{p},\sigma)}{\partial\sigma\,\partial \mathrm{p}} & -\frac{\partial^2\delta_\varphi^*(\mathrm{p},\sigma)}{\partial\sigma^2} \end{pmatrix}^{-1}$$

Thus, the bayes estimates parameter p and σ under SELF are given by,

$$\hat{\mathrm{p}}_{\mathrm{TK}}=\sqrt{\frac{|\Sigma_{\mathrm{p}}^*|}{|\Sigma|}}\,\mathrm{e}^{\mathrm{n}[\delta_{\mathrm{p}}^*(\hat{\mathrm{p}}_{\delta^*},\hat{\sigma}_{\delta^*})} - \mathrm{e}^{\mathrm{n}\delta_{\mathrm{p}}(\hat{\mathrm{p}},\hat{\sigma})]},$$

$$\hat{\sigma}_{\mathrm{TK}}=\sqrt{\frac{|\Sigma_{\sigma}^*|}{|\Sigma|}}\,\mathrm{e}^{\mathrm{n}[\delta_{\sigma}^*(\hat{\mathrm{p}}_{\delta^*},\hat{\sigma}_{\delta^*})} - \mathrm{e}^{\mathrm{n}\delta_{\sigma}(\hat{\mathrm{p}},\hat{\sigma})]} \text{ respectively.}$$

The Bayes estimator of survival and hazard function under SELF, we consider φ (p, σ) = S(t) and h(t) and the estimators are:

$$\hat{\mathrm{S}}(\mathrm{t})_{\mathrm{TK}}=\sqrt{\frac{|\Sigma^*{}_{\mathrm{S(t)}}|}{|\Sigma|}}\,\mathrm{e}^{\mathrm{n}\delta^*{}_{\mathrm{S(t)}}(\hat{\mathrm{p}}_{\delta^*},\hat{\sigma}_{\delta^*})} - \mathrm{e}^{\mathrm{n}\delta^*{}_{\mathrm{S(t)}}(\hat{\mathrm{p}},\hat{\sigma})} \text{ and}$$

$$\hat{\mathrm{h}}(\mathrm{t})_{\mathrm{TK}}=\sqrt{\frac{|\Sigma^*{}_{\mathrm{h(t)}}|}{|\Sigma|}}\,\mathrm{e}^{\mathrm{n}\delta^*{}_{\mathrm{h(t)}}(\hat{\mathrm{p}}_{\delta^*},\hat{\sigma}_{\delta^*})} - \mathrm{e}^{\mathrm{n}\delta^*{}_{\mathrm{h(t)}}(\hat{\mathrm{p}},\hat{\sigma})} \text{respectively.}$$

3.2 MCMC Method

The MCMC method is the important technique to get samples from posterior distributions. MH algorithm Metropolis et al. (1953) & Hastings (1970) is one of the Markov chain simulation methods that are used to generate samples from the joint posterior distribution whenever the posterior distribution cannot be reduced to standard form. It is also discussed in Robert and Casella (2004). The full conditionals for p and σ can be given as,

$$\pi(p|\,\sigma, x) \;\propto p^{n+a_1-1}\sigma^{-np}e^{-(b_1 p)}e^{-\Sigma\left(\frac{x_i}{\sigma}\right)^p}\prod_{i=1}^{n}\frac{x_i{}^{p-1}}{\left\{1-\frac{e^{-\left(\frac{x_i}{\sigma}\right)^p}}{(1+\alpha^{\beta})}\right\}^2} \tag{14}$$

$$\pi(\sigma\,|\,p, x) \;\propto \sigma^{-np+a_2-1}e^{-(b_2\sigma)}e^{-\Sigma\left(\frac{x_i}{\sigma}\right)^p}\prod_{i=1}^{n}\left\{1-\frac{e^{-\left(\frac{x_i}{\sigma}\right)^p}}{(1+\alpha^{\beta})}\right\}^{-2} \tag{15}$$

The conditional densities given in equations (14) and (15) are not from known distributions and the MH algorithm is used to generate posterior samples from the given full conditional densities. The steps employed to generate posterior samples are as follows:

(i). Start with an initial guess, say ($p^{(0)}$, $\sigma^{(0)}$)

(ii). Set k = 1

(iii). Generate $p_c^{(k)}$, from the proposal normal density N($p^{(k-1)}$,1).

(iv). Generate u from Uniform (0,1).

(v). Now, compute $r(p^{(k)})|\, p^{(k-1)}) = \min\left\{\frac{\pi_p(p_c^{(k)}|\text{data})}{\pi_p(p^{(k-1)}|\text{data})}, 1\right\}$

(vi). If $u \le r$, set $p(k) = p_c^{(k)}$ with acceptance probability r otherwise, set $p(k) = p^{(k-1)}$.

(vii). Generate $\sigma^{(k)}$using similar steps with proposal normal density N($\sigma^{(k-1)}$,1).

(viii). Set k = k +1.

(ix). Repeat step (iii) to (viii) for N times to get the MCMC samples of p and σ as $p^{(k)}$ and $\sigma^{(k)}$respectively for k = 1,2, …, N.

Next, we discarded first $N_0 = 20\%$ of the N samples as burn-in-period and obtained independent samples from the stationary distribution of the Markov chain. To minimize the autocorrelation, we may alternatively use the thinning interval, which involves discarding all samples except for the j-th generated ones.

Now, the bayes estimator of $\varphi(p, \sigma)$ is defined as

$$\hat{\varphi}_{MC}(p, \sigma) = \frac{1}{N-N_0}\sum_{k=N_0+1}^{N}\hat{\varphi}\left(p^{(k)}, \sigma^{(k)}\right)$$

Therefore, taking $\varphi(p, \sigma)$ = p and σ, Bayes estimates of p and σ under SELF are given by respectively,

$$\hat{p}_{MC} = \frac{1}{N-N_0}\sum_{k=N_0+1}^{N}p^{(k)},$$

$$\hat{\sigma}_{MC} = \frac{1}{N - N_0} \sum_{k=N_0+1}^{N} \sigma^{(k)},$$

The bayes estimates of SF and HF are obtained as, respectively

$$\hat{S}(t)_{MC} = \frac{1}{N - N_0} \sum_{k=N_0+1}^{N} \frac{\alpha^{\beta} e^{-\left(\frac{x}{\sigma^{(k)}}\right)^{p^{(k)}}}}{\alpha^{\beta} + 1 - e^{-\left(\frac{x}{\sigma^{(k)}}\right)^{p^{(k)}}}}; \qquad t > 0$$

$$\hat{h}(t)_{MC} = \frac{1}{N - N_0} \sum_{k=N_0+1}^{N} \frac{\left(1 + \alpha^{\beta}\right) \frac{p^{(k)}}{\sigma^{(k)p^{(k)}}} x^{p^{(k)}-1}}{\alpha^{\beta} + 1 - e^{-\left(\frac{x}{\sigma^{(k)}}\right)^{p^{(k)}}}}; \qquad t > 0$$

3.3 HPD Credible Interval Estimation

To get the interval estimate, the HPD credible intervals of parameters are constructed using the algorithm proposed by Chen and Shao (1999). Let $p_{(1)} < p_{(2)} < \cdots < p_{(N-N_0)1}$ be the ordered values of the generated MCMC samples of p in the previous subsection. The 100(1-ξ)% HPD credible interval for p is,

$$\left(p_{(k)}, p_{(k+[(1-\xi)(N-N_0)])}\right),$$

where k is so chosen that

$$p_{(k+[(1-\xi)(N-N_0)])} - p_{(k)} = \min_{1 \le i \le (N-N_0)} \left(p_{(i+[(1-\xi)(N-N_0)])} - p_{(i)}\right); k = 1,2, \dots, (N - N_0)$$

here, [x] is the largest integer less than or equal to x. Similarly, the HPD credible intervals for parameter σ can also be obtained.

4. Simulation Study

A simulation study is performed to evaluate the effectiveness and performance of proposed Bayes estimators and ML estimates.

The random sample is generated from the M-W distribution using the inverse cdf transformation as,

$$x_i, = \sigma\left(\ln\left(1 - \frac{u_i}{1 + \alpha^{\beta}}\right) - \ln(1 - u_i)\right)^{\frac{1}{p}}$$

where $u_i \sim U(0, 1)$.

The 1,000 random samples are generated using the above approach for four sample sizes 25, 50, 75 and 100. We fixed the value of parameters (α, β) at (0.5,1) (0.5,2) and (2,2). To compare the various estimates three sets of true values of parameters are taken as (p, σ) =(0.6, 1.0), (p, σ) = (1.5, 1.5) and (p, σ) = (2.0, 2.0). The ML estimates are obtained for each sample size. For Bayesian computation method, the values of hyperparameters are taken as $a_1 = b_1 = a_2 = b_2 =$ 0.0001 using non-informative prior (P1). In case of informative prior (P2), the hyper parameters values are taken as $a_1 = 3$, $b_1 = 5$, $a_2 = 1$, $b_2 = 1$ when (p, σ) = (0.6, 1.0), $a_1 = 3$, b_1

= 2, a_2 = 3, b_2 = 2 when (p, σ) = (1.5, 1.5) and a_1 = 2, b_1 = 1, a_2 = 2, b_2 = 1 when (p, σ) = (2, 2). These values are so chosen that the prior mean is exactly equal to the true values of the corresponding parameter. The bayes estimates are obtained using discussed approximation techniques. The posterior samples for the parameters p and σ of size 10,000 are generated and 2,000 observations are discarded as burn-in period in MCMC method. The mean squared error (MSE) of estimators is calculated for comparison of different estimates. Further, average length (AL) of ACI and HPD intervals and coverage probability (CP) are calculated as interval estimates.

The MSEs of ML and Bayes estimates of p and σ are given in Tables 1,3 and 6. Table 2,4 and 6 contain the CP and AL of the ACI and HPD credible intervals of parameters p and σ for different values of α and β. It is observed from these results that:

1. The MSE of both estimates tends to decrease as sample size increases. The Bayes estimates have the lesser MSE than ML estimates when they include prior information about the parameters. Also, bayes estimates evalutaed for P2 give better results than bayes estimates for P1.
2. The bayes estimates obtained using MH algorithm with informative priors have the lesser MSE than ML estimates and bayes estimates using TK method and hence provide best results among all estimates.
3. The AL of ACI/HPD credible intervals narrows down as sample size increase in all cases. Also, credible intervals of parameter p have shorter AL than ACIs and thus provide better results whereas they are better only in case of informative prior for parameter σ.
4. For ML estimation, the CP attain the prescribed confidence levels satisfactorily and for Bayesian estimation, it attains the nominal level almost in all cases.

   Therefore, we recommand to use bayes estimates to get point and interval estimates for estimation purposes as they are good in terms of MSE.

Table 1. MSE of estimators when α = 0.5 and β = 1.0

(p, σ)	**n**	$\hat{p}$					$\hat{\sigma}$				
		MLE	**TK**		**MH**		**MLE**	**TK**		**MH**	
			P1	**P2**	**P1**	**P2**		**P1**	**P2**	**P1**	**P2**
0.6,1	25	0.0118	0.0119	0.0094	0.0101	0.0076	0.2105	0.3894	0.1852	0.3597	0.1668
	50	0.0048	0.0047	0.0045	0.0042	0.0035	0.1139	0.1134	0.1068	0.1323	0.0985
	75	0.0029	0.0028	0.0027	0.0025	0.0024	0.0726	0.0864	0.0716	0.0813	0.0602
	100	0.0021	0.0020	0.0019	0.0020	0.0018	0.0547	0.0632	0.0542	0.0586	0.0481
1.5,1.5	25	0.0737	0.0622	0.0527	0.0478	0.0417	0.0729	0.0694	0.0494	0.0845	0.0649
	50	0.0301	0.0276	0.0256	0.0249	0.0241	0.0366	0.0363	0.0346	0.0390	0.0348
	75	0.0183	0.0171	0.0163	0.0157	0.0142	0.0259	0.0214	0.0250	0.0252	0.0234
	100	0.0131	0.0124	0.0120	0.0130	0.0117	0.0192	0.0195	0.0189	0.0188	0.0178
2.0,2.0	25	0.1311	0.1104	0.0989	0.1240	0.0810	0.0815	0.0771	0.0673	0.0815	0.0635
	50	0.0536	0.0539	0.0531	0.0516	0.0472	0.0399	0.0377	0.0349	0.0383	0.0347
	75	0.0326	0.0304	0.0295	0.0318	0.0274	0.0261	0.0253	0.0227	0.0203	0.0201
	100	0.0233	0.0223	0.0217	0.0231	0.0215	0.0193	0.0189	0.0186	0.0186	0.0183

Table 2. AL and CP of interval estimates when α = 0.5 and β=1.0

(p, σ)	n	Method	$\hat{p}$ AL	$\hat{p}$ CP	$\hat{\sigma}$ AL	$\hat{\sigma}$ CP
0.6,1	25	MLE	0.3618	0.945	1.7729	0.881
		P1	0.3483	0.948	1.8017	0.938
		P2	0.3325	0.962	1.7266	0.942
	50	MLE	0.2501	0.942	1.2216	0.910
		P1	0.2379	0.973	1.2377	0.964
		P2	0.2319	0.938	1.2124	0.957
	75	MLE	0.2015	0.959	1.0002	0.917
		P1	0.1918	0.933	1.0166	0.905
		P2	0.1864	0.948	0.9784	0.938
	100	MLE	0.1733	0.949	0.8689	0.932
		P1	0.1610	0.933	0.8581	0.944
		P2	0.1607	0.944	0.8426	0.956
1.5,1.5	25	MLE	0.9146	0.939	1.0149	0.927
		P1	0.8009	0.936	1.0488	0.953
		P2	0.7506	0.943	1.0143	0.943
	50	MLE	0.6245	0.953	0.7198	0.933
		P1	0.5992	0.938	0.7291	0.958
		P2	0.5834	0.945	0.7191	0.963
	75	MLE	0.5030	0.944	0.5890	0.933
		P1	0.4784	0.928	0.5928	0.962
		P2	0.4777	0.954	0.5852	0.916
	100	MLE	0.4326	0.946	0.5125	0.936
		P1	0.4136	0.930	0.5065	0.932
		P2	0.4115	0.956	0.5022	0.945
2.0,2.0	25	MLE	1.2190	0.955	0.9886	0.898
		P1	1.1983	0.948	1.1074	0.952
		P2	1.1269	0.950	1.0662	0.968
	50	MLE	0.8302	0.948	0.7193	0.936
		P1	0.8058	0.948	0.7530	0.934
		P2	0.7949	0.954	0.7444	0.960
	75	MLE	0.6714	0.954	0.5895	0.935
		P1	0.6383	0.930	0.6058	0.954
		P2	0.6279	0.948	0.6058	0.967
	100	MLE	0.5781	0.947	0.5114	0.947
		P1	0.5559	0.952	0.5059	0.942
		P2	0.5519	0.956	0.5023	0.924

Table 3. MSE of estimators when α = 0.5 and β =2.0

(p, σ)	n	$\hat{p}$ MLE	$\hat{p}$ TK P1	$\hat{p}$ TK P2	$\hat{p}$ MH P1	$\hat{p}$ MH P2	$\hat{\sigma}$ MLE	$\hat{\sigma}$ TK P1	$\hat{\sigma}$ TK P2	$\hat{\sigma}$ MH P1	$\hat{\sigma}$ MH P2
0.6,1	25	0.0103	0.0099	0.0080	0.0091	0.0072	0.3217	0.6704	0.1869	0.8631	0.1849
	50	0.0042	0.0043	0.0036	0.0040	0.0029	0.1647	0.2186	0.1236	0.2472	0.1147
	75	0.0029	0.0027	0.0026	0.0026	0.0022	0.1006	0.1239	0.0973	0.1365	0.0792
	100	0.0021	0.0019	0.0018	0.0016	0.0014	0.0797	0.0878	0.0660	0.0769	0.0467
1.5,1.5	25	0.0641	0.0612	0.0561	0.0627	0.0518	0.1175	0.1300	0.0927	0.1240	0.0755
	50	0.0264	0.0269	0.0218	0.0246	0.0215	0.0542	0.0536	0.0499	0.0527	0.0408
	75	0.0181	0.0167	0.0158	0.0134	0.0124	0.0336	0.0359	0.0330	0.0365	0.0243
	100	0.0134	0.0121	0.0116	0.0096	0.0071	0.0238	0.0267	0.0231	0.0237	0.0222
2.0,2.0	25	0.1169	0.1085	0.0868	0.0821	0.0799	0.1153	0.1225	0.0967	0.1200	0.0869
	50	0.0519	0.0479	0.0442	0.0394	0.0364	0.0567	0.0548	0.0527	0.0521	0.0502
	75	0.0284	0.0297	0.0259	0.0285	0.0245	0.0368	0.0353	0.0317	0.0385	0.0278
	100	0.0234	0.0202	0.0197	0.0201	0.0185	0.0282	0.0291	0.0280	0.0288	0.0246

Table 4. AL and CP of interval estimates when α = 0.5 and β=2.0

(p, σ)	n	Method	$\hat{p}$ AL	$\hat{p}$ CP	$\hat{\sigma}$ AL	$\hat{\sigma}$ CP
0.6,1	25	MLE	0.3635	0.941	2.1273	0.846
		P1	0.3549	0.937	3.2874	0.952
		P2	0.3178	0.952	1.8587	0.954
	50	MLE	0.2464	0.967	1.4910	0.885
		P1	0.2406	0.948	1.7115	0.939
		P2	0.2303	0.980	1.4759	0.960
	75	MLE	0.1999	0.942	1.1921	0.893
		P1	0.1940	0.926	1.3055	0.956
		P2	0.1894	0.960	1.1606	0.946
	100	MLE	0.1703	0.950	1.0608	0.909
		P1	0.1665	0.960	1.0339	0.953
		P2	0.1336	0.950	0.8278	0.946
1.5,1.5	25	MLE	0.9064	0.938	1.1996	0.875
		P1	0.8763	0.936	1.1965	0.940
		P2	0.8325	0.942	1.1800	0.956
	50	MLE	0.6140	0.941	0.8596	0.906

		P1	0.5958	0.936	0.8597	0.940
		P2	0.5814	0.932	0.8590	0.928
	75	MLE	0.4963	0.944	0.6967	0.930
		P1	0.4779	0.944	0.6968	0.938
		P2	0.4684	0.940	0.6923	0.928
	100	MLE	0.4267	0.957	0.6174	0.942
		P1	0.4182	0.936	0.6162	0.948
		P2	0.4116	0.948	0.6127	0.946
2.0,2.0	25	MLE	1.2043	0.949	1.1914	0.889
		P1	1.1316	0.960	1.1962	0.947
		P2	1.1275	0.966	1.1872	0.944
	50	MLE	0.8290	0.950	0.8480	0.923
		P1	0.7981	0.935	0.8152	0.940
		P2	0.7720	0.953	0.8001	0.933
	75	MLE	0.6607	0.949	0.7222	0.935
		P1	0.6342	0.921	0.7133	0.940
		P2	0.6221	0.953	0.7003	0.973
	100	MLE	0.5683	0.952	0.6098	0.945
		P1	0.5497	0.927	0.6079	0.947

Table 5. MSE of estimators when α = 2.0 and β = 2.0

(p, σ)	n	$\hat{p}$					$\hat{\sigma}$				
		MLE	TK		MH		MLE	TK		MH	
			P1	P2	P1	P2		P1	P2	P1	P2
0.6,1	25	0.0128	0.0110	0.0106	0.0112	0.0076	0.1455	0.1794	0.1247	0.1507	0.0828
	50	0.0053	0.0049	0.0048	0.0045	0.0041	0.0722	0.0800	0.0679	0.0790	0.0659
	75	0.0032	0.0030	0.0029	0.0030	0.0029	0.0462	0.0496	0.0448	0.0492	0.0411
	100	0.0023	0.0022	0.0021	0.0022	0.0021	0.0344	0.0363	0.0337	0.0321	0.0298
1.5,1.5	25	0.0799	0.0685	0.0575	0.0688	0.0552	0.0494	0.0510	0.0446	0.0516	0.0418
	50	0.0332	0.0307	0.0284	0.0273	0.0267	0.0251	0.0254	0.0238	0.0237	0.0212
	75	0.0201	0.0190	0.0180	0.0209	0.0179	0.0162	0.0163	0.0156	0.0159	0.0154
	100	0.0145	0.0139	0.0134	0.0141	0.0130	0.0121	0.0122	0.0118	0.0120	0.0114
2.0,2.0	25	0.1278	0.1217	0.0977	0.1173	0.0963	0.0441	0.0505	0.0430	0.0546	0.0428
	50	0.0570	0.0546	0.0495	0.0591	0.0471	0.0236	0.0253	0.0234	0.0236	0.0216
	75	0.0325	0.0337	0.0299	0.0228	0.0221	0.0147	0.0162	0.0146	0.0135	0.0134
	100	0.0269	0.0247	0.0241	0.0231	0.0107	0.0119	0.0122	0.0118	0.0113	0.0107

Table 6. AL and CP of interval estimates when α = 2.0 and β=2.0

(p, σ)	n	Method	$\hat{p}$		$\hat{\sigma}$	
			AL	CP	AL	CP
0.6,1	25	MLE	0.3892	0.955	1.3653	0.8900
		P1	0.3679	0.948	1.4059	0.8960
		P2	0.3337	0.920	1.1727	0.9160
	50	MLE	0.2634	0.962	0.9864	0.9310
		P1	0.2549	0.944	0.9852	0.9040
		P2	0.2503	0.946	0.9326	0.9430
	75	MLE	0.2129	0.948	0.8035	0.9320
		P1	0.2045	0.934	0.7876	0.8940
		P2	0.1972	0.946	0.7717	0.9390
	100	MLE	0.1828	0.949	0.7063	0.9340
		P1	0.1767	0.948	0.7011	0.9080
		P2	0.1758	0.960	0.6745	0.9640
1.5,1.5	25	MLE	0.9571	0.953	0.8274	0.938
		P1	0.8905	0.958	0.8321	0.960
		P2	0.8820	0.938	0.8263	0.926
	50	MLE	0.6623	0.965	0.5829	0.924
		P1	0.6308	0.946	0.5821	0.934
		P2	0.6187	0.936	0.5765	0.950
	75	MLE	0.5328	0.942	0.4822	0.929
		P1	0.5054	0.912	0.4759	0.912

		P2	0.5028	0.938	0.4671	0.912
	100	MLE	0.4576	0.953	0.4169	0.939
		P1	0.4366	0.936	0.4049	0.930
		P2	0.4299	0.940	0.4044	0.939
2.0,2.0	25	MLE	1.2794	0.954	0.8748	0.908
		P1	1.2425	0.956	0.8696	0.948
		P2	1.0380	0.958	0.8643	0.942
	50	MLE	0.8769	0.943	0.5970	0.936
		P1	0.8379	0.932	0.5896	0.956
		P2	0.7800	0.945	0.5801	0.955
	75	MLE	0.7062	0.949	0.4845	0.943
		P1	0.6760	0.953	0.4811	0.946
		P2	0.6623	0.945	0.4810	0.936
	100	MLE	0.6127	0.947	0.4178	0.931
		P1	0.5678	0.944	0.4049	0.950
		P2	0.5505	0.946	0.4043	0.953

5. Real Data Analysis

In this section, we demonstrate the estimation methods discussed earlier using a dataset. The observations are the survival times (in days) of guinea pigs that were given different doses of tubercle bacilli. These records are taken from Bjerkedal (1960).

For the particular experimental regimen (6.6), a total of 72 survival time observations were recorded, as listed below:

12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

First, we provide the descriptive statistics for this data in Table 7 to summarize its key statistical properties. It suggests that the data is right-skewed, have a long tail toward higher survival times and greater variability. The behaviour of data has been examined using several graphical plots such as box plot, TTT plot and Kaplan Meier (KM) Survival plot.

Next, we fit several statistical distributions to the dataset to determine the most appropriate model for the data. Specifically, we consider the M-W distribution, weibull distribution (WD), gamma distribution (GD), the exponentiated-exponential distribution (EED) and the Modified-Exponential (M-E) distribution. The negative Log-Likelihood (-lnL), information criterians namely AIC, BIC, CAIC and HQIC are used to assess the fit. The goodness of fit statistics, Kolmogorov-Smirnov (K-S) Statistic and p-value are evaluated to compare the empirical distribution of the data with the fitted models. The best model for the data is selected based on the lowest values of -lnL, AIC, BIC, CAIC, HQIC and K-S statistic, along with the highest p-value. The MLEs and K-S statistic are reported in Table 8 and Information criterion are given in Table 9. The results show that M-W distribution is the best model for this data.

In Table 10, all estimates with respective 95% ACI/HPD intervals of the parameters are reported. The Bayes estimates of parameters are computed with non-informative priors using both approximation techniques. In M-H algoritham, the Markov chain is generated taking M = 1,00,000 and first M_0 = 20,000 values are discarded as burn-in-period and every 10th value in generated samples is taken as iid observation.

5.1 Data Visualization

In this subsection, we draw box plot, scaled TTT plot and KM survival plot of the data. The Box plot shows the positively skewed nature of the data, TTT plot suggests a unimodal hazard function and survival plot shows the survival probabilities, see figure (1). These figures suggest to model the data using the M-W distribution. The fitting plot of the M-W distribution to the empirical distribution of the data is given in figure (2) and it suggests that proposed model fits data very well.

For Bayesian estimation using MCMC method, the convergence for the stationary distributions of markov chains is verified using graphical diagnostic tools like trace plot, auto correlation function (ACF) plot and kernel density plots. Figure (3) shows the trace, ACF and kernel density plots for the parameters obtained from the MCMC sampling process. The trace plots of the chains indicate a random scatter about the mean value (represented by solid line) and reflects good convergence of the MCMC process. The ACF plots display the chains for both p and σ show very low autocorrelations and generated samples are nearly independent and representative of the true posterior distribution. The kernal density plots show that the conditional marginal distributions of the parameters are appear to be symmetrical and unimodal i.e. mean can be taken as the best estimate for the parameters.

Table 7. Descriptive statistics

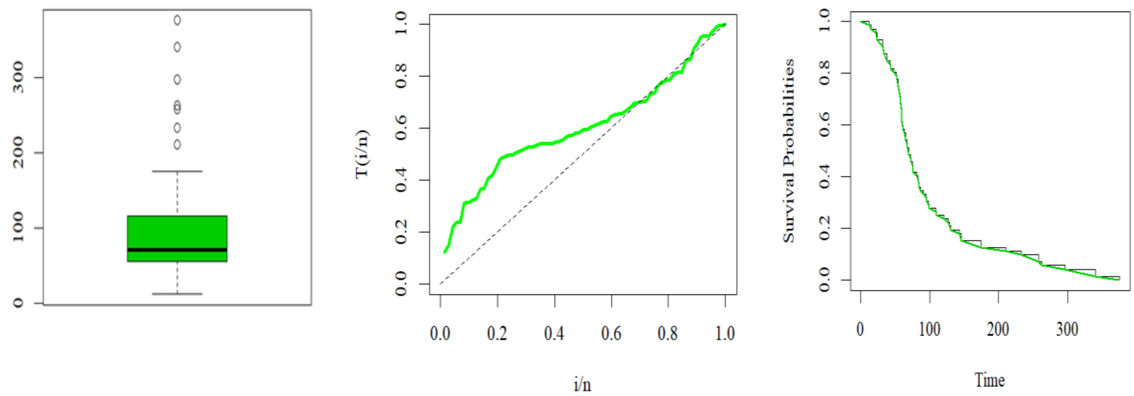
Min	Q1	Medium	Mean	Q3	Max	Skewness	Kurtosis	Sd
12.00	54.75	70.00	99.82	112.75	376.00	1.79624	2.61444	81.11795

Table 8. MLEs, Neg log-likelihood and Goodness-of-fit test Statistics

Distribution	MLEs	-ln L	K-S	p-value
M-W (α=0.1, β=1.0)	**2.199594, 237.828567**	**390.4254**	**0.1009**	**0.4553**
WD	1.3925, 110.3530	397.1479	0.14551	0.09479
GD	2.0812, 0.0209	394.2476	0.1381	0.128
EED	2.4842, 0.0170	393.1106	0.13244	0.1599
M-E	3.5642, 6.1615, 0.010	403.4448	0.21213	0.0031

Table 9. Information Criterion				
Distribution	**AIC**	**BIC**	**CAIC**	**HQIC**
M-W	**784.8509**	**789.4042**	**785.0248**	**786.6635**
WD	798.2958	802.8491	798.4697	800.1085
GD	792.4952	797.0485	792.6691	794.3079
EED	790.2212	794.7745	790.3951	792.0339
M-E	812.8897	819.7197	813.2426	815.6087

Table 10. Estimates of parameters, survival and hazard functions with 95% ACI/HPD credible intervals (in bracket) for survival times (in days) of guinea pigs				
Method	$\hat{\mathbf{p}}$	$\hat{\boldsymbol{\sigma}}$	**S(t)**	**h(t)**
MLE	2.1996 (1.8591, 2.5401)	237.8286 (190.8541, 284.8030)	0.5642	0.0142
Bayesian (MCMC)	2.168671 (1.8451, 2.4966)	244.645244 (198.4372, 297.0932)	0.5702	0.0138
Bayesian (TK)	2.168176	237.8286	0.5705	0.0116



(a) *(b)* *(c)*

Figure 1. *(a) Box plot, (b) TTT plot and (c) KM Survival Plot for the data*

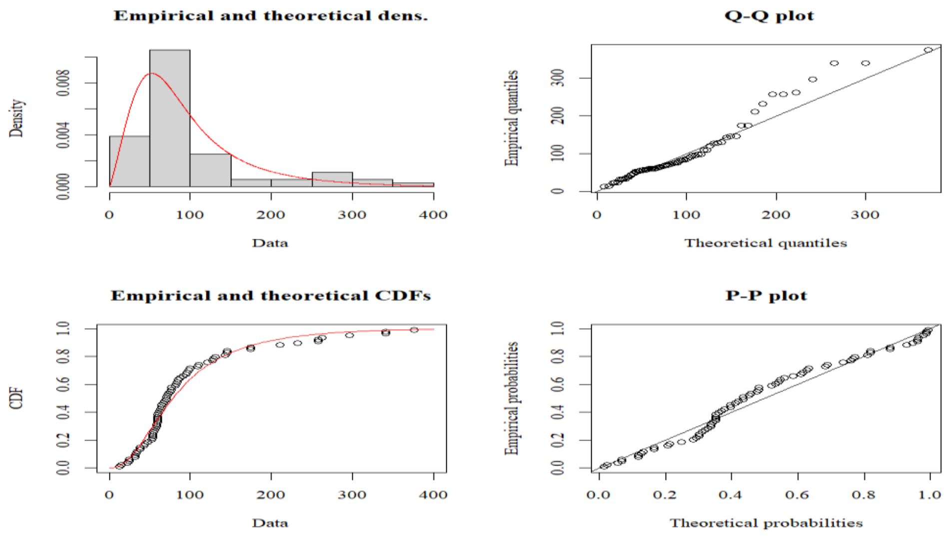


Figure 2. *Fitting plot of M-W distribution to empirical data*

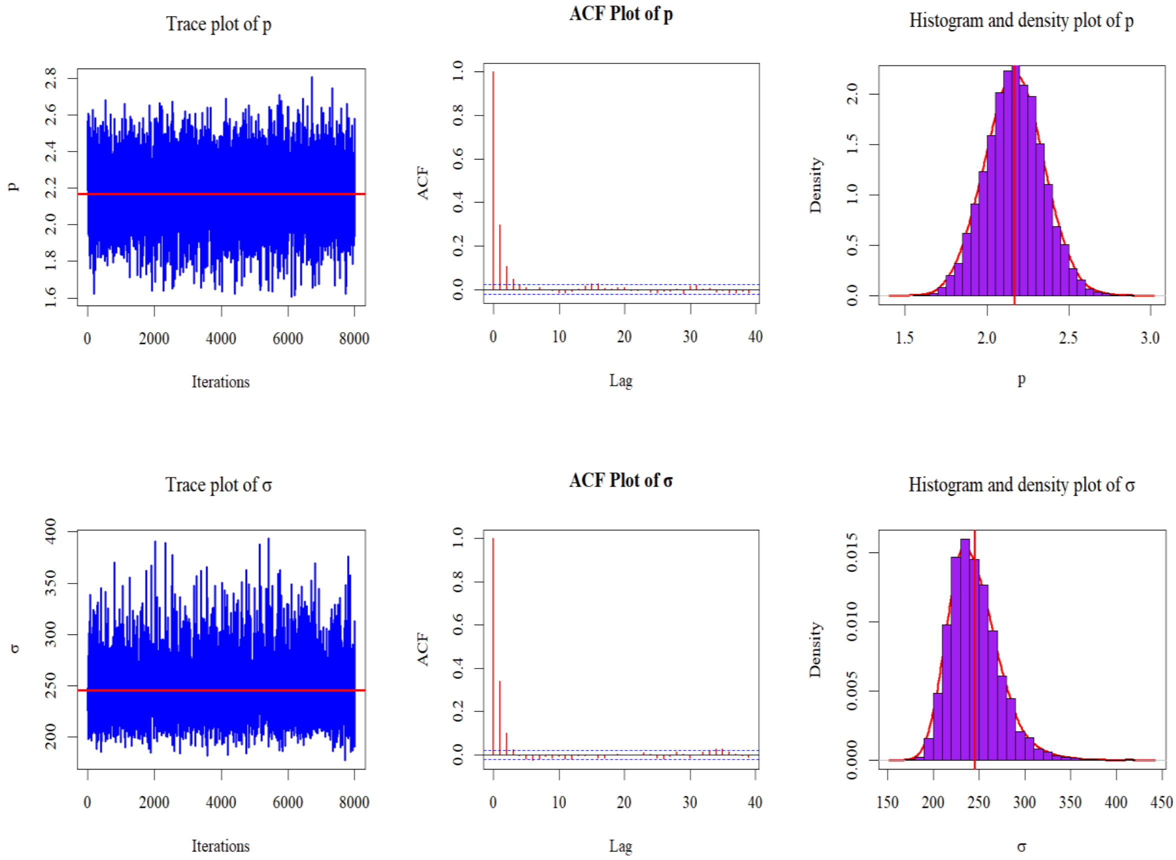


Figure 3. *Trace plot, ACF plot and histogram with kernel density plots of the parameters*

6. Concluding Remarks

This article studies the Bayesian estimation of the parameters and reliability functions of M-W distribution when parameters α and β are known. The ML estimates and ACIs are obtained of unknown parameters. For Bayes estimates, the prior belief is considered by the independent gamma informative and non-informative priors. Bayes estimates are computed under SELF using TK method and MCMC methods. The M-H algorithm is used to generate MCMC samples of parameters and to compute credible intervals. The performance of both estimators is compared by a monte carlo simulation study. The study shows that bayes estimates using informative priors obtained from MH algorithm have minimum MSE than ML and the Bayes estimates using TK method under both informative and non-informative priors. The HPD credible interval estimates also provided better results than ACI under informative priors. Next, we analysed a real set of observations to implement the discussed estimation methods that provided satisfactory results under Bayesian approach. Thus, we recommend to use Bayes estimation with some prior information or a suitable non-informative prior for more efficient results.

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